

# Transverse force on a vortex in lattice models of superfluids

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**Abstract** – The Letter addressees vortex dynamics for the lattice models of superfluids in the continuous approximation. The continuous approximation restores translational invariance absent in the original lattice model, but the theory is not Galilean invariant. As a result, calculation of the two transverse forces on the vortex, Magnus force and Lorentz force, requires the analysis of two balances, for the genuine momentum of particles in the lattice (Magnus force) and for the quasimomentum (Lorentz force) similar to that in the Bloch theory of particles in the periodic potential. While the developed theory yields the same Lorentz force, which was well known before, a new general expression for the Magnus force was obtained. The theory demonstrates how a small Magnus force emerges in the Josephson-junction array if the particle-hole symmetry is broken. The continuous approximation for the Bose–Hubbard model close to the superfluid–insulator transition was developed, which was used for calculation of the Magnus force. The theory shows that there is an area in the phase diagram for the Bose–Hubbard model, where the Magnus force has an inverse sign with respect to that which is expected from the sign of velocity circulation.

**Introduction.** – The transverse force on a vortex in superfluids (neutral and charged) is debated during many decades and has been a topic of reviews and books [1–4]. In a continuous superfluid at  $T = 0$ , which is identical to a perfect fluid in classical hydrodynamics, the balance of forces on a vortex is

$$\mathbf{F}_M + \mathbf{F}_L = mn[(\mathbf{v}_L - \mathbf{v}_s) \times \boldsymbol{\kappa}] = \mathbf{F}_{ext}, \quad (1)$$

where the Magnus force  $\mathbf{F}_M$  is proportional to the vortex velocity  $\mathbf{v}_L$  and the Lorentz force  $\mathbf{F}_L$  is proportional to the superfluid velocity  $\mathbf{v}_s = \frac{\hbar}{m} \nabla \varphi$  determined by the phase  $\varphi$  of the order parameter wave function, and the external force  $\mathbf{F}_{ext}$  combines all other forces on the vortex, e.g., pinning and friction forces. Here  $n$  is the density of particles with mass  $m$ , and  $\boldsymbol{\kappa}$  is the vector parallel to the vortex axis with its modulus equal to the circulation quantum  $\kappa = h/m$ . The united transverse force  $\mathbf{F}_M + \mathbf{F}_L$  depends only on the relative velocity  $\mathbf{v}_L - \mathbf{v}_s$  as required by the Galilean invariance.

In lattice models of superfluids the Galilean invariance is absent, and the value of the Magnus force was under scrutiny. The most known lattice model of the superfluid is the Josephson junction array. Usually they studied the theory of vortex motion in the array using the continu-

ous approximation. These studies have not revealed any Magnus force normal to the vortex velocity [5]. Moreover, there have been experimental evidences of the ballistic vortex motion in the Josephson junction array [6], which is possible only in the absence of the Magnus force. There was a lot of theoretical works trying to explain this important feature of the Josephson junction array, mostly suggesting that quantum effects can provide a finite Magnus force. Meanwhile, in the classical theory of the Josephson junction array they usually assumed the particle-hole symmetry, which forbids the Hall effect and the Magnus force in the model [7]. However, this symmetry is not exact, and its violation allows the existence of a finite Magnus force. In superconductors the Magnus force determines the Hall effect, and the presence or the absence of this force means the presence or the absence of the Hall effect.

Intensive experimental and theoretical investigations of Bose-condensed cold atoms attracted an interest to another lattice model of a superfluid: the Bose–Hubbard model [8]. The periodic structure of potential wells for bosons, which leads to the Bose–Hubbard model in the tight-binding limit, is realised for cold-atom BEC in the experiments with optical lattices [9]. Recently Lindner *et al.* [10] and Huber and Lindner [11] addressed the prob-

lem of the Magnus force in the Bose–Hubbard model and revealed that close to the superfluid–insulator transition the force changes its sign as happens in Fermi superfluids when the sign of the carrier charge changes its sign.

The Letter presents the analysis of the transverse force on the vortex in a lattice, which is approximated by a continuous model with effective parameters determined by properties of the lattice. The forces are determined from the momentum balance. The absence of the Galilean invariance makes necessary to analyse two momentum balances: for genuine momentum and for quasimomentum similar to that in the Bloch band theory for particles in a periodic potential. This yields the general expression for the Magnus force in the absence of the Galilean invariance. The expression allowed to calculate the Magnus force in the Bose–Hubbard model close to the superfluid–insulator transition after deriving the continuous theory for the superfluid order parameter near this transition.

**Vortex dynamics in the continuous approximation for the lattice superfluid.** – The continuous approximation for lattice superfluids gives the phenomenological theory reminding the usual hydrodynamics. Generally enough this approximation corresponds to the following Lagrangian:

$$L = -\hbar n \dot{\varphi} - \frac{\hbar^2 \tilde{n}}{2m} (\nabla \varphi)^2 - E_c(n). \quad (2)$$

where  $E_c(n)$  is the energy of a resting liquid which depends only on  $n$ . For simplicity we consider the 2D problem, where  $n$  is the particle number per unit area. The Hamiltonian (energy) for this Lagrangian is

$$H = \frac{\partial L}{\partial \dot{\varphi}} \dot{\varphi} - L = \frac{\hbar^2 \tilde{n}}{2m} (\nabla \varphi)^2 + E_c(n). \quad (3)$$

Despite similarity to hydrodynamics of the perfect fluid, there is an essential difference. The continuous approximation for the lattice model restores translational invariance but not Galilean invariance. The latter is absent since the effective density  $\tilde{n}$ , which characterises stiffness of the phase field, is different from the genuine particle density  $n$ , and is much less than  $n$  if the lattice nodes are weakly connected.

Let us discuss the conservation laws, which follow from Noether’s theorem. The gauge invariance provides the conservation law for charge (particle number):

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\varphi}} + \nabla_k \left( \frac{\partial L}{\partial \nabla_k \varphi} \right) = 0, \quad (4)$$

which in fact is the continuity equation (the first Hamilton equation) for the density  $n$ :

$$\frac{dn}{dt} = -\frac{\hbar}{m} \nabla_k (\tilde{n} \nabla_l \varphi). \quad (5)$$

The second Hamilton equation is

$$\hbar \frac{\partial \varphi}{\partial t} = -\frac{\partial H}{\partial n} = -\mu - \frac{\hbar^2}{2m} \frac{d\tilde{n}}{dn} (\nabla_j \varphi)^2, \quad (6)$$

where  $\mu = \partial E_c(n)/\partial n$  is the chemical potential of the liquid at rest.

The translational invariance provides the conservation law

$$\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{\varphi}} \nabla_k \varphi \right) + \nabla_l \left( \frac{\partial L}{\partial \nabla_k \varphi} \nabla_l \varphi - L \delta_{ij} \right) = 0. \quad (7)$$

This is the conservation law

$$\frac{\partial g_k}{\partial t} + \nabla_l \Pi_{kl} = 0, \quad (8)$$

for the current

$$\mathbf{g} = -\frac{\partial L}{\partial \dot{\varphi}} \nabla \varphi = \hbar n \nabla \varphi. \quad (9)$$

Here the momentum–flux tensor is

$$\begin{aligned} \Pi_{kl} &= \frac{\hbar^2}{m} \tilde{n} \nabla_k \varphi \nabla_l \varphi \\ &+ \left[ P + \frac{\hbar^2}{2m} \left( \frac{d\tilde{n}}{dn} n - \tilde{n} \right) (\nabla_j \varphi)^2 \right] \delta_{kl}, \end{aligned} \quad (10)$$

and the pressure is determined from the  $T = 0$  thermodynamic Gibbs–Duhem relation  $dP = n d\mu$ .

The current  $\mathbf{g}$  is not a genuine mass current. The genuine mass current is that appears in the continuity equation (5) and is the density of the genuine momentum:

$$\mathbf{j} = -\frac{m}{\hbar} \frac{\partial L}{\partial \nabla \varphi} = \hbar \tilde{n} \nabla \varphi. \quad (11)$$

So Noether’s theorem does not provides the conservation law for the momentum in a non-Galilean liquid. Nevertheless, the momentum conservation law for the Lagrangian (2) takes place though it is derived not from Noether’s theorem but directly from explicit equations of motion:

$$\frac{\partial j_k}{\partial t} + \nabla_l \tilde{\Pi}_{kl} = 0, \quad (12)$$

where the momentum–flux tensor is

$$\tilde{\Pi}_{kl} = \frac{\hbar^2}{m} \frac{d\tilde{n}}{dn} \tilde{n} \nabla_k \varphi \nabla_l \varphi + \tilde{P} \delta_{kl}, \quad (13)$$

and the partial pressure  $\tilde{P}$  is determined from the relation  $d\tilde{P} = \tilde{n} d\mu$ . These equations reduce to those in hydrodynamics of a Galilean invariant liquid when  $\tilde{n} = n$ . In contrast to the conservation law eq. (8) following from Noether’s theorem, the conservation law (12) for the genuine mass current is approximate and neglects higher than second order terms in phase gradients.

For better understanding of the physical difference between the two currents  $\mathbf{g}$  and  $\mathbf{j}$ , let us consider particles in a periodic potential, which can be also described by our continuous model. The wave functions of particles are described by Bloch functions:

$$\psi(\mathbf{r}) = u_n(\mathbf{r}, \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}}, \quad (14)$$

where  $u_n(\mathbf{r}, \mathbf{k})$  is the periodic function and the wave vector  $\mathbf{k}$  determines the quasimomentum  $\hbar\mathbf{k}$ . The quasimomentum differs from the genuine momentum of the quantum state equal to the average of the momentum operator:

$$\mathbf{p} = -i\hbar \int \psi(\mathbf{r})^* \frac{\partial \psi(\mathbf{r})}{\partial \mathbf{r}} d\mathbf{r} = \hbar\mathbf{k} - i\hbar \int u(\mathbf{r})^* \frac{\partial u(\mathbf{r})}{\partial \mathbf{r}} d\mathbf{r}. \quad (15)$$

Calculating the band energy  $E(\mathbf{k})$  in the  $\mathbf{k}\mathbf{p}$  approximation for small  $k$ , i.e., at the band bottom (the energy minimum at  $k = 0$ ) one obtains that

$$\mathbf{p} = m\mathbf{v}_g = \frac{m}{m^*} \hbar\mathbf{k}, \quad (16)$$

where

$$\mathbf{v}_g = \frac{dE(\mathbf{k})}{\hbar d\mathbf{k}} \approx \frac{d^2 E(\mathbf{k})}{\hbar d\mathbf{k}^2} \mathbf{k} = \frac{\hbar\mathbf{k}}{m^*}, \quad (17)$$

is the group velocity and  $m^*$  is the effective mass. The vector  $\mathbf{p}$  is a genuine momentum indeed since particles with bare mass  $m$  propagate in space with the group velocity. Assuming that particles are bosons, which condense in a single Bloch state with density  $n$ , and connecting the wave vector  $\mathbf{k} = \nabla\varphi$  with the phase  $\varphi$ , the density of the genuine momentum, i.e. the genuine mass current is,  $\mathbf{j} = n\mathbf{p}$  coincide with that given by eq. (11) with  $\tilde{n} = nm/m^*$ . On the other hand, the current  $\mathbf{g} = n\hbar\mathbf{k}$  is the quasimomentum density. It is well known from the solid state physics [12] that an external force on a particle in the energy band determines time variation of the quasimomentum but not the genuine momentum:

$$\hbar \frac{d\mathbf{k}}{dt} = m^* \frac{d\mathbf{v}_g}{dt} = \mathbf{f}. \quad (18)$$

In the absence of Umklapp processes the total quasimomentum is also a conserved quantity, and the conservation law for the Bose-condensate in a single Bloch state is given by eq. (8). The physical meaning of the effective second Newton law (18) is that only the part  $d\mathbf{p}/dt$  of the whole momentum  $\hbar d\mathbf{k}/dt$  brought to the system by the external force is transferred to the momentum of particles moving in the periodic potential. The rest part  $d(\hbar\mathbf{k} - \mathbf{p})/dt$  is transferred to the system supporting the periodic potential, i.e., to the lattice.

For derivation of forces on the vortex one must calculate the momentum flux through a cylindric surface surrounding the vortex line [1, 7]. We argue that the Lorentz and the Magnus force must be derived from the balance of different momenta: the quasimomentum for the former and the genuine momentum for the latter. Deriving the Lorentz force one can assume that the vortex is at rest in the laboratory system connected with the lattice. Solving eq. (6) for the time-independent phase  $\varphi$  one obtains the quadratic in  $\nabla\varphi$  correction to the chemical potential (Bernoulli's effect):

$$\mu' = -\frac{\hbar^2}{2m} \frac{d\tilde{n}}{dn} (\nabla_j \varphi)^2. \quad (19)$$

Then the momentum-flux tensor (10) becomes

$$\Pi_{kl} = \frac{\hbar^2}{m} \tilde{n} \nabla_k \phi \nabla_l \varphi + [P_0 - \tilde{n} (\nabla_j \varphi)^2] \delta_{kl}, \quad (20)$$

where  $P_0$  is a constant pressure in the absence of any velocity field. The components of the Lorentz force are given by the integral over the cylinder around the vortex:  $F_{Li} = \oint \Pi_{kl} dS_l$ . Taking into account that the phase gradient  $\nabla\varphi = \nabla\varphi_v + \nabla\varphi_j$  consists of the gradient  $\nabla\varphi_v = [\hat{z} \times \mathbf{r}]/2\pi r^2$  induced by the vortex line and the gradient  $\nabla\varphi_j = \mathbf{j}/\hbar\tilde{n}$  produced by the transport current, the integration yields the Lorentz force

$$\mathbf{F}_L = -[\mathbf{j} \times \boldsymbol{\kappa}] = -m\tilde{n}[\mathbf{v}_s \times \boldsymbol{\kappa}]. \quad (21)$$

This expression for the Lorentz force is well known but now we derived it from the quasimomentum balance. This is because the force is a transfer of momentum from the transport velocity field to the vortex. But any variation of the transport velocity must be accompanied by the momentum transfer to or from the lattice. So as in the effective second Newton law (18), the force is a time variation of quasimomentum but not of genuine momentum.

The derivation of the Magnus force requires the balance of the genuine momentum. It is more convenient to consider this balance in the coordinate frame moving with the vortex where variables are time-independent. In the frame moving with the velocity  $\mathbf{w}$  the energy  $E'$  is connected with the energy  $E$  in the laboratory frame by the relation  $E' = E - \mathbf{j} \cdot \mathbf{w}$  (we ignore quadratic in  $w$  terms). Then according to the Hamiltonian (3) written in the laboratory frame

$$E' = \frac{\hbar^2 \tilde{n}}{2m} (\nabla\varphi')^2 + E_c(n), \quad (22)$$

where the phase gradient in the moving frame is  $\nabla\varphi' = \nabla\varphi - m\mathbf{w}/\hbar$ . This does not differ from the Hamiltonian in the laboratory frame, and one can use the momentum-flux tensor (13) replacing  $\varphi$  by  $\varphi'$ . The pressure variation is determined from the Bernoulli law as before. Deriving the Magnus force one may ignore the superfluid current in the laboratory frame, i.e.  $\nabla\varphi = 0$  and replace  $\mathbf{w}$  by  $\mathbf{v}_L$ . Then calculating the Magnus force components  $F_{Mi} = \oint \tilde{\Pi}_{kl} dS_l$  one obtains the Magnus force

$$\mathbf{F}_M = \frac{d\tilde{n}}{dn} \tilde{n} m [\mathbf{v}_L \times \boldsymbol{\kappa}]. \quad (23)$$

The force appears due to convection of the vortex-related momentum  $\hbar\tilde{n}\nabla\varphi_v$  into the area of the momentum balance by the supercurrent. Since the circular velocity field  $\frac{\hbar}{m}\nabla\varphi_v$  is fixed an arrival of a particle into the balance area may change the vortex-related momentum only via variation of  $\tilde{n}$  and does not require accompanying momentum transfer to the lattice. This is why the Magnus force is determined from the balance of the genuine momentum and is proportional to  $d\tilde{n}/dn$ .

In the Josephson junction array the current between two nodes of the lattice is determined by the Josephson energy  $E_J \cos(\varphi_1 - \varphi_2)$ , where  $\varphi_1$  and  $\varphi_2$  are the phases of the two nodes. In the continuous limit this yields  $\tilde{n} = mE_J/\hbar^2$ . The particle-hole symmetry requires that  $E_J$  and  $\tilde{n}$  do not depend on the average density  $n$ , and the Magnus force vanishes in agreement with the symmetry of this model.

Including the Lorentz and the Magnus force into the balance of forces on the vortex, like that given by eq. (1) for the Galilean invariant liquid, one sees that the total transverse force  $\mathbf{F}_M + \mathbf{F}_L$  is proportional to the velocity  $(d\tilde{n}/dn)\mathbf{v}_L - \mathbf{v}_s$ , but not to the relative velocity  $\mathbf{v}_L - \mathbf{v}_s$ .

The role of the first term  $-\hbar n\dot{\varphi}$  in the Lagrangian (2), which is called the Wess–Zumino term, was widely discussed in the literature [4]. The arguments were about whether the total liquid density  $n$  must be replaced by some other density. It is evident that adding any constant  $C$  to the density  $n$  in the Wess–Zumino term does not affect the dynamical equations. However the role of the Wess–Zumino term changes after transition from the continuous model in terms of fields to the reduced description in terms of the vortex coordinates  $\mathbf{r}_L(x_L, y_L)$ . For the Wess–Zumino term this leads to substitution of the phase field  $\varphi_v(\mathbf{r}, \mathbf{r}_L)$  for moving vortex into the term and its integration over the whole space. Bearing in mind that  $\dot{\varphi}_v = -(\mathbf{v}_L \cdot \nabla)\varphi_v$ , the Wess–Zumino term becomes

$$L_{WZ} = -\hbar(n + C)\mathbf{v}_L \cdot [\hat{\mathbf{z}} \times \mathbf{r}_L]. \quad (24)$$

Varying the total Lagrangian of the vortex with respect to  $\mathbf{r}_L(t)$ , one obtains the equation of vortex motion with the effective Magnus force  $\propto (n + C)$ . So the constant  $C$  does matter for the value of the Magnus force. Sometimes it is argued that the contribution  $\propto C$  is of topological origin and can be found from the topological analysis [4]. On the other hand, we think that there is no general principle, which dictates the charge in the Wess–Zumino term. Not the undefined Wess–Zumino term determines what and whether the transverse force is, but vice versa, one must derive the transverse force from dynamical equations and only after this one knows what Wess–Zumino term in the vortex Lagrangian should be. In particular, in the Galilean translation invariant liquid one obtains from the momentum-conservation law that the amplitude of the Magnus force is proportional to the total density. Then only the latter enters the Wess–Zumino term and  $C = 0$ . On the other hand, if in the Fermi superfluid the core bound states are intensively scattered by impurities, the Kopnin–Kravtsov force [2] fully compensates the Magnus force. Then the vortex Lagrangian does not contain the Wess–Zumino term at all.

#### Vortex dynamics in the Bose–Hubbard model.

– The Hamiltonian of the Bose–Hubbard model [8] for a lattice with distance  $a$  between nodes is

$$\mathcal{H} = -J \sum_{i,j} \hat{b}_i^\dagger \hat{b}_j + \frac{U}{2} \sum_i \hat{N}_i(\hat{N}_i - 1) - \mu \sum_i \hat{N}_i. \quad (25)$$

Here  $\mu$  is the chemical potential, the operators  $\hat{b}_i$  and  $\hat{b}_i^\dagger$  are the operators of annihilation and creation of a boson at the  $i$ th lattice node, and  $\hat{N}_i = \hat{b}_i^\dagger \hat{b}_i$  is the particle number operator at the same node. The first sum is over neighbouring lattice nodes  $i$  and  $j$ .

In the superfluid phase with large numbers of particles  $N_i$  all operator fields can be replaced by the classical fields in the spirit of the Bogolyubov theory:

$$\hat{b}_i \rightarrow \sqrt{N_i} e^{i\varphi_i}, \quad \hat{b}_i^\dagger \rightarrow \sqrt{N_i} e^{-i\varphi_i}, \quad (26)$$

where  $\varphi_i$  is the phase at the  $i$ th node. After transition to the continuous approach one obtains the Hamiltonian (3), where<sup>1</sup>

$$n = \frac{N}{a^2}, \quad \tilde{n} = \frac{mz_0 J a^2}{\hbar^2} n, \quad E_c(n) = \frac{U a^2}{2} n^2 - \mu n. \quad (27)$$

Here  $z_0$  is the number of nearest neighbours equal to 4 in the quadratic lattice. Thus the effective density  $\tilde{n}$  is proportional to the total density  $n$ , as in the case of the condensate of bosons in a single Bloch states close to the bottom of the energy band considered in the previous section. This is not accidental since in the limit of weak interaction  $U$  the hopping term  $\sim J$  in the Hamiltonian also yields the energy band, which is described by the effective mass  $m^*$  near its bottom, and the ratio  $\tilde{n}/n$  is equal to the ratio of masses  $m/m^*$ . According to derivation of the previous section, the Magnus force [see eq. (23)] is by the factor  $\tilde{n}^2/n^2 = m^2 J^2 a^4/\hbar^4$  smaller than in the Galilean-invariant superfluid. This factor must be small because of weak Josephson links between the potential wells.

It is known that when the energy  $J$  of the internode hopping decreases, the phase transition from superfluid to Mott insulator must occur [8]. In the limit  $z_0 J/U \rightarrow 0$  when the hopping term  $\propto J$  can be ignored the eigenstates are given by Fock states  $|\Psi_N\rangle = |N\rangle$  with fixed number  $N$  of particles at any node. At growing  $J$  the transition line can be found in the mean-field approximation [13]. In this approximation the hopping term is taking into account by introducing the mean field equal to the average value of the annihilation operator its complex conjugate being the average value of the creation operator:

$$\langle \hat{b}_i \rangle = \psi_i = |\psi| e^{i\varphi_i}, \quad \langle \hat{b}_i^\dagger \rangle = \psi_i^* = |\psi| e^{-i\varphi_i}. \quad (28)$$

It is assumed that only the phase but not the modulus of the order parameter  $\psi$  varies from node to node. In contrast to eq. (26), in general  $|\psi|^2$  is not equal to  $N$  and must be determined from the condition of self-consistency (see below). Introducing the mean field one reduces the problem to the single-node problem with the Hamiltonian

$$\mathcal{H}_s = -zJ(\hat{b}_i^\dagger \psi + \psi^* \hat{b}_i) + \frac{U}{2} \hat{N}_i(\hat{N}_i - 1) - \mu \hat{N}_i. \quad (29)$$

<sup>1</sup>We assume that there is the same number of particles at all nodes and write  $N_i$  without the subscript  $i$ .

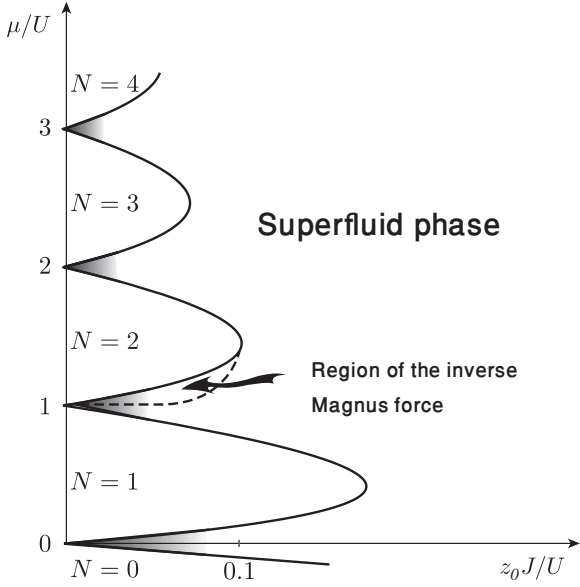


Fig. 1: The phase diagram of the Bose-Hubbard model. The Mott insulator phase occupies lobes corresponding to fixed integer numbers  $N$  of bosons. The shaded beaks of the superfluid phase, which penetrate between insulator lobes, are analysed in the text. The dash line separates the region with the inverse Magnus force from the rest of the superfluid phase. The line is schematic since it was not really calculated. The region of the inverse Magnus force exists under any lobe but is shown only for the beak between the  $N = 1$  and  $N = 2$  lobes.

Here

$$z = \sum_j e^{i(\varphi_j - \varphi_i)} \quad (30)$$

reduces to the number  $z_0$  of nearest neighbours in the uniform state with the constant phase at all nodes.

The many-node wave function is a product of the single-node wave functions. Calculating the energy of the original Hamiltonian (25) for this wave function and minimising it with respect to  $\psi$  one obtains the self-consistency equation, which determines  $\psi$ . Following this approach [13] one obtains the phase diagram shown in Fig. 1. The Mott-insulator phases with fixed numbers  $N$  of particles per node occupy the interiors of lobes at small  $z_0 J/U$ .

We are interested to know what vortex dynamics is expected close to the phase transition at minimal values of  $J$ , i.e. at beaks of the superfluid phase between lobes, which are shaded in Fig. 1. Here the mean-field approximation is simplified by the fact that only two states with  $N$  and  $N + 1$  particles interplay at the superfluid phase in the beak corresponding to the given  $N$ . This is because at  $\mu = NU$  these two states have the same energy, whereas all other states are separated by the gap on the order of the high energy  $U$ . So for a beak between the lobes corresponding to Mott insulators with the number of bosons per node  $N$  and  $N + 1$  we look a solution in the form of a superposition of two Fock states:

$$|\Psi_N\rangle = f_N|N\rangle + f_{N+1}|N+1\rangle. \quad (31)$$

This wave function is an eigenfunction of the single-node Hamiltonian (29) if

$$f_N = \sqrt{\frac{\sqrt{\frac{\mu'^2}{4} + z^2 J^2 (N+1)} |\psi|^2 - \frac{\mu'}{2}}{2\sqrt{\frac{\mu'^2}{4} + z^2 J^2 (N+1)} |\psi|^2}},$$

$$f_{N+1} = \sqrt{\frac{\sqrt{\frac{\mu'^2}{4} + z^2 J^2 (N+1)} |\psi|^2 + \frac{\mu'}{2}}{2\sqrt{\frac{\mu'^2}{4} + z^2 J^2 (N+1)} |\psi|^2}} e^{i\varphi}. \quad (32)$$

The energy of this eigenstate is

$$\epsilon_N = -\mu' \left( N + \frac{1}{2} \right) \pm \sqrt{\frac{\mu'^2}{4} + z^2 J^2 (N+1) |\psi|^2}, \quad (33)$$

where  $\mu' = \mu - Un$ , and one should choose the lower sign - for the ground state. The average number of particles in the node is a function of  $\mu'$ :

$$\langle \hat{N} \rangle = N + \frac{1}{2} + \frac{\mu'}{2\sqrt{\mu'^2 + 4z^2 J^2 (n+1) |\psi|^2}}. \quad (34)$$

The self-consistency equation follows either from the minimisation of the total energy with respect to  $\psi$  or from the condition that  $\psi$  is the average value of the operator  $\hat{a}$ :

$$\psi = \langle \hat{b} \rangle = \frac{zJ(N+1)}{2\sqrt{\frac{\mu'^2}{4} + z^2 J^2 (N+1) |\psi|^2}} \psi. \quad (35)$$

A non-trivial (i.e., non-zero) solution of this equation is

$$|\psi|^2 = \frac{N+1}{4} - \frac{\mu'^2}{4z^2 J^2 (N+1)}. \quad (36)$$

The eigenvalue  $\epsilon_N$  determines the average value of the Hamiltonian (25), which is the Gibbs thermodynamic potential of the grand canonical ensemble per one node:

$$G_N = zJ|\psi|^2 + \epsilon_N. \quad (37)$$

Since the Hamilton equations are formulated for the canonically conjugated variables “phase-particle number” it is useful to go from the grand canonical ensemble with the Gibbs potential being a function of  $\mu$  to the canonical ensemble where the energy density is a function of the particle number density  $n$ . Then the energy per node is

$$E_N = G_N + \mu N = \frac{UN^2}{2} + UNN_e + zJ \left( |\psi|^2 - 2\sqrt{N+1} \sqrt{\frac{1}{4} - N_e^2} |\psi| \right), \quad (38)$$

where

$$N_e = \langle \hat{N} \rangle - N - \frac{1}{2}. \quad (39)$$

The energy has a minimum at

$$|\psi|^2 = (N+1) \left( \frac{1}{4} - N_e^2 \right). \quad (40)$$



As is expected in the case of the second-order phase transition,  $\psi$  vanishes at the phase transition lines, where  $N_e = \pm \frac{1}{2}$  and the number of particles reaches  $N$  at the lower border and  $N + 1$  at the upper one. In contrast to the Landau-Lifshitz theory of the second order transitions, there is no analytic expansion in  $\psi$  near the critical temperature because of the term linear in  $|\psi|$ .

In order to derive parameters of the continuous model discussed in the previous section, let us consider the effect of slow phase variation from node to node. Assuming that  $\psi_i = |\psi|e^{i\mathbf{k}\cdot\mathbf{r}_i}$  where  $\mathbf{r}_i$  is the position vector of the  $i$ th node, one obtains for the square lattice with the number  $z_0 = 4$  of nearest neighbours:

$$z = 2 \cos(k_x a) + 2 \cos(k_y a) \approx 4 - k^2 a^2. \quad (41)$$

Bearing in mind the correspondence of the wave vector  $\mathbf{k}$  to the gradient operator  $\nabla$  in the configurational space one obtains in the continuum limit for small  $\mathbf{k}$  the Hamiltonian (3) with

$$\begin{aligned} \tilde{n} &= \frac{2m}{\hbar^2} J(N+1) \left( \frac{1}{4} - n_e^2 a^4 \right), \\ E_c(n) &= 4J(N+1) \left( \frac{1}{4} - n_e^2 a^4 \right), \end{aligned} \quad (42)$$

where the effective density  $n_e = N_e/a^2$  differs from the total density  $n = (N + \frac{1}{2})/a^2 + n_e$  by a constant and constant terms in the energy were ignored. This allows to find the density dependent factor in the expression (23) for the Magnus force:

$$\frac{d\tilde{n}}{dn} \tilde{n} = -\frac{8m^2}{\hbar^4} J^2 a^4 (N+1)^2 n_e \left( \frac{1}{4} - n_e^2 a^4 \right). \quad (43)$$

A remarkable feature of the Magnus force in the beaks of the superfluid phase is that its sign is not a sign of the total charge proportional to the total density of the liquid but is a sign of the effective density  $n_e$ . So in the upper halves of the beaks shown in Fig. 1 the Magnus force has an inverse sign with respect to the total charge. If the liquid were electrically charged this would mean an inverse sign of the Hall conductivity. The regions of the inverse Magnus force neighbour any insulator lobe from below, where  $n_e$  is positive. Since at upper borders of the lobes  $n_e$  is negative, the line  $n_e = 0$  must end somewhere at the border of the lobe. In Fig. 1 it is shown by a dashed line for the beak between the  $N = 1$  and  $N = 2$  lobes. The connection of the Hall conductivity with the sign of  $n_e$  was pointed out by Lindner *et al.* [10] for the lowest beak  $N = 0$ , i.e., for the superposition of the states  $N = 0$  and  $N = 1$  at the node, which characterises the model of hard-core bosons (the limit  $U \rightarrow \infty$ ). However they obtained the Magnus force essentially different from eq. (23) and having a jump at the line where the force changes sign.

**Conclusion.** — We derived the transverse (Magnus and Lorentz) forces on the vortex from the conservation

laws in the continuous approximation for lattice models of superfluids. It is important that the two forces are obtained from two different conservation laws, one for the genuine momentum of particles in the lattice (Magnus force), another for the quasimomentum (Lorentz force) similar to that in the Bloch band theory. The calculated Magnus force vanishes for the Josephson junction array as required by the particle-hole symmetry usually assumed in the theory of the array. The analysis of the Bose-Hubbard model in the continuous approximation yielded the expression for the Magnus force, which had an inverse sign at some area close to the phase superfluid-insulator transition. Sign inversion was revealed earlier for hard-core bosons by a different method [10, 11] but from the expression for the Magnus force different from ours.

Though our analysis addressed ideal strictly periodical lattices, the results of the analysis are expected to be valid also for superfluids in random potentials (e.g., superfluids in porous media or on disordered substates), where broken translational and Galilean invariance must also weaken the Magnus force.

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